III. *On the Contact of Curves.* By WILLIAM SPOTTISWOODE, *M.A.*, *F.R.S.*

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LET U=0 be the equation to the curve with which the curve V=0 has an "*m*-pointic contact;" in other words, let the curves U and V have *m* consecutive points in common. The degree of contact of which V is capable is equal to the number of independent constants contained in its equation; *i. e.* to the number of terms in its complete expression, less one. Thus if *n* be the degree of V, the degree of contact will be $=\frac{(n+1)(n+2)}{1.2}-1$.

If the curves U and V have only a single point in common, then the only conditions are

$$U=0,\ V=0. \quad \ldots \quad (1.)$$

If they have two consecutive points in common, then beside (1.) we have also

$$\left.\begin{array}{l}\frac{\partial U}{dx}dx+\frac{\partial U}{dy}dy+\frac{\partial U}{dz}dz=0,\\ \frac{\partial V}{dx}dx+\frac{\partial V}{dy}dy+\frac{\partial V}{dz}dz=0,\end{array}\right\} \quad \ldots \quad (2.)$$

which, as is well known, lead to the conditions

$$\left.\begin{array}{l}\frac{\partial U}{dx}:\frac{\partial U}{dy}:\frac{\partial U}{dz}\\ =\frac{\partial V}{dx}:\frac{\partial V}{dy}:\frac{\partial V}{dz};\end{array}\right\} \quad \ldots \quad (3.)$$

and if V be linear, $=lx+my+nz$, (3.) will suffice to determine the ratios $l:m:n$, and fix the position of the tangent V. The conditions (3.) may be expressed by a single equation thus: if α, β, γ be any arbitrary quantities, then the equation

$$\left|\begin{array}{ccc}\alpha & \beta & \gamma\\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz}\\ \frac{\partial V}{dx} & \frac{\partial V}{dy} & \frac{\partial V}{dz}\end{array}\right|=0, \quad \ldots \quad (4.)$$

considered as identical in α, β, γ, may be regarded as an expression for the required conditions.

The developed form of (4.) is

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{dV}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial V}{dz}=0; \quad \ldots \quad (5.)$$

so that, writing

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{\partial}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial}{dz}=\square, \quad . \quad . \quad (6.)$$

the effect of the operation $\square$ upon V is equivalent to the elimination of dx, dy, dz, from (1.) and their differentials (2.), and is expressed by the equation

$$\square V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

If the curves have three consecutive points in common, we have, in addition to former conditions,

$$\left.\begin{aligned}\frac{\partial U}{dx}d^2x+\frac{\partial U}{dy}d^2y+\frac{\partial U}{dz}d^2z+\frac{\partial^2 U}{dx^2}dx^2+\frac{\partial^2 U}{dy^2}dy^2+\frac{\partial^2 U}{dz^2}dz^2+2\left(\frac{\partial^2 U}{dydz}dydz+\frac{\partial^2 U}{dzdx}dzdx+\frac{\partial^2 U}{dxdy}dxdy\right)=0,\\ \frac{\partial V}{dx}d^2x+\frac{\partial V}{dy}d^2y+\frac{\partial V}{dz}d^2z+\frac{\partial^2 V}{dx^2}dx^2+\frac{\partial^2 V}{dy^2}dy^2+\frac{\partial^2 V}{dz^2}dz^2+2\left(\frac{\partial^2 V}{dydz}dydz+\frac{\partial^2 V}{dzdx}dzdx+\frac{\partial^2 V}{dxdy}dxdy\right)=0,\end{aligned}\right\} \quad (8.)$$

from which dx, dy, dz, d^2x, d^2y, d^2z are to be eliminated by the help of (1.) and (2.), or (3.). Instead, however, of differentiating (2.), we may differentiate (3.), or, what is the same thing, (4.); and we shall then have an expression free from d^2x, d^2y, d^2z.

The results of the elimination may, moreover, in this case be grouped into a single expression. Writing for convenience

$$\frac{\partial U}{dy}\frac{\partial}{dz}-\frac{\partial U}{dz}\frac{\partial}{dy}=D_1, \quad \frac{\partial U}{dz}\frac{\partial}{dx}-\frac{\partial U}{dx}\frac{\partial}{dz}=D_2, \quad \frac{\partial U}{dx}\frac{\partial}{dy}-\frac{\partial U}{dy}\frac{\partial}{dx}=D_3, \quad . \quad . \quad . \quad (9.)$$

(3.) may be expressed by any two of the equations

$$D_1V=0, \quad D_2V=0, \quad D_3V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Differentiating these and combining their differentials with the first of (2.), we may form the quadratic system,

$$\left.\begin{matrix} D_1^2V\ \ =0 & D_1D_2V=0 & D_1D_3V=0 \\ D_2D_1V=0 & D_2^2V\ \ =0 & D_2D_3V=0 \\ D_3D_1V=0 & D_3D_2V=0 & D_3^2V\ \ =0, \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

all of which are comprised in the one equation

$$\square^2V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

And generally, by a similar train of reasoning, the constants of a curve V, having an m-pointic contact with U, will be determined by the equations

$$V=0, \quad \square V=0, \quad .. \ \square^m V=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

and if

$$\left.\begin{aligned} V&=(a, b, ..\, k\between x, y, z)^n \\ &=a\frac{\partial V}{da}+b\frac{\partial V}{db}+..\ k\frac{\partial V}{dk}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

then

$$a : b : \ldots k$$

$$= \left\| \begin{matrix} \frac{\partial V}{da} & \frac{\partial V}{db} & \ldots & \frac{\partial V}{dk} \\ \square\frac{\partial V}{da} & \square\frac{\partial V}{db} & \ldots & \square\frac{\partial V}{dk} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \square^{m-1}\frac{\partial V}{da} & \square^{m-1}\frac{\partial V}{db} & - & \square^{m-1}\frac{\partial V}{dk} \end{matrix} \right\| \quad \ldots \ldots \quad (15.)$$

The equations (11.) are, however, not all independent. In the first place, as in (10.) any one of the three equations is a consequence of the other two, so in (12.) any one column or row is a consequence of the other two. Secondly, if any pair of binary combinations of the Ds, *e. g.* D_1D_2V and D_2D_1V, be developed, they will, with the help of (10.) or (3.), be found identical. This reduces the system (12.) to three independent conditions, of which

$$D_1^2V=0, \quad D_1D_2V=0, \quad D_2^2V=0 \quad \ldots \ldots \quad (16.)$$

is a type; as it should do. These three equations will suffice to determine the three independent constants of V, when it is capable of a 3-pointic contact, and no more, with U.

The actual calculations may be considerably simplified by using, instead of (13.), the following system,

$$V=0, \quad DV=0, \quad D^2V=0$$

(where, as before, D stands for any one of the symbols D_1, D_2, D_3). Of this system (16.) is a consequence.

By way of example, we have for the ordinary tangent

$$V=ax+by+cz=0,$$
$$DV=aDx+bDy+cDz=0,$$

whence

$$a : b : c$$

$$\left\| \begin{matrix} x & y & z \\ Dx & Dy & Dz \end{matrix} \right\| \quad \ldots \ldots \quad (17.)$$

$$=yDz-zDy : zDx-xDz : xDy-yDx$$

$$=y\frac{\partial U}{dy}+z\frac{\partial U}{dz} : -x\frac{\partial U}{dy} : -x\frac{\partial U}{dz}$$

$$=\frac{\partial U}{dx} : \frac{\partial U}{dy} : \frac{\partial U}{dz}. \quad \ldots \ldots \quad (18.)$$

The equation of the circle of curvature may be put in the form

$$hr^2+2(ayz+bzx+cxy)=0;$$

to which are to be added,

$$hD\,r^2+2(aD\,yz+bD\,zx+cD\,xy)=0,$$
$$hD^2r^2+2(aD^2yz+bD^2zx+cD^2xy)=0,$$

whence

$$h : a : b : c$$

$$\left\| \begin{matrix} r^2 & 2yz & 2zx & 2xy \\ \mathrm{D}r^2 & 2\mathrm{D}yz & 2\mathrm{D}zx & 2\mathrm{D}xy \\ \mathrm{D}^2r^2 & 2\mathrm{D}^2yz & 2\mathrm{D}^2zx & 2\mathrm{D}^2xy ; \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

also

$$\mathrm{D}r^2 = 2\left(z\frac{\partial \mathrm{U}}{dy} - y\frac{\partial \mathrm{U}}{dz}\right),$$

$$\mathrm{D}^2r^2 = 2\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2 + \left(\frac{\partial \mathrm{U}}{dz}\right)^2\right\} + 2\left(z\mathrm{D}\frac{\partial \mathrm{U}}{dy} - y\mathrm{D}\frac{\partial \mathrm{U}}{dz}\right),$$

$$\mathrm{D}yz = y\frac{\partial \mathrm{U}}{dy} - z\frac{\partial \mathrm{U}}{dz},$$

$$\mathrm{D}^2yz = -2\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} + y\mathrm{D}\frac{\partial \mathrm{U}}{dy} - z\mathrm{D}\frac{\partial \mathrm{U}}{dz},$$

$$\mathrm{D}zx = x\frac{\partial \mathrm{U}}{dy} \quad \mathrm{D}^2zx = x\mathrm{D}\frac{\partial \mathrm{U}}{dy},$$

$$\mathrm{D}xy = -x\frac{\partial \mathrm{U}}{dz} \quad \mathrm{D}^2xy = -x\mathrm{D}\frac{\partial \mathrm{U}}{dz}.$$

Moreover the minors formed from the first two rows of the matrix, and from the columns below written, have the following values:—

$$r^2 \;, \; 2yz = x\left\{-(y^2-z^2)\frac{\partial \mathrm{U}}{dx} + xy\frac{\partial \mathrm{U}}{dy} - xz\frac{\partial \mathrm{U}}{dz}\right\},$$

$$r^2 \;, \; 2zx = x\left\{ -yx\frac{\partial \mathrm{U}}{dx} - (z^2-x^2)\frac{\partial \mathrm{U}}{dy} + yz\frac{\partial \mathrm{U}}{dz}\right\},$$

$$r^2 \;, \; 2xy = x\left\{ zx\frac{\partial \mathrm{U}}{dx} - zy\frac{\partial \mathrm{U}}{dy} - (x^2-y^2)\frac{\partial \mathrm{U}}{dz}\right\},$$

$$2zx, \; 2xy = 2x^3\frac{\partial \mathrm{U}}{dx},$$

$$2xy, \; 2yz = 2xy^2\frac{\partial \mathrm{U}}{dy},$$

$$2yz, \; 2zx = 2xz^2\frac{\partial \mathrm{U}}{dz};$$

whence, finally,

$$h : a : b : c$$

$$= 4\frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} + \frac{2xyz}{(n-1)^2}\mathrm{H}$$

$$: 2\frac{\partial \mathrm{U}}{dx}\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2 + \left(\frac{\partial \mathrm{U}}{dz}\right)^2\right\} + (\;\; x^2-y^2-z^2)\frac{x}{(n-1)^2}\mathrm{H}$$

$$: 2\frac{\partial \mathrm{U}}{dy}\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2 + \left(\frac{\partial \mathrm{U}}{dx}\right)^2\right\} + (-x^2+y^2-z^2)\frac{y}{(n-1)^2}\mathrm{H}$$

$$: 2\frac{\partial \mathrm{U}}{dz}\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2 + \left(\frac{\partial \mathrm{U}}{dy}\right)^2\right\} + (-x^2-y^2+z^2)\frac{z}{(n-1)^2}\mathrm{H}, \quad . \quad . \quad . \quad . \quad (20.)$$

where H is the Hessian of U.

Proceeding by the same principles, we shall have merely to replace □ by D in (15.), and there will result a system of expressions for the constants of the curve V having an m-pointic contact with a given curve U.

The following method of reduction may sometimes be used with advantage. Let $\xi, \eta, \zeta, \ldots \phi, \chi, \psi$ stand for the powers and products of the variables forming the various terms of V. Then the coefficients of the curve V having an m-pointic contact with U will be proportional to the determinants of which the following is a type:

$$\begin{matrix} D^{m-1}\xi & D^{m-1}\eta & \ldots & D^{m-1}\psi \\ D^{m-2}\xi & D^{m-2}\eta & \ldots & D^{m-2}\psi \\ \vdots & \vdots & & \vdots \\ \xi & \eta & \ldots & \psi. \end{matrix}$$

But this

$$=-\Sigma D^{m-2}\xi \begin{vmatrix} D^{m-1}\eta & D^{m-1}\zeta & \ldots & D^{m-1}\psi \\ D^{m-3}\eta & D^{m-3}\zeta & \ldots & D^{m-3}\psi \\ \vdots & \vdots & & \vdots \\ \eta & \zeta & \ldots & \psi \end{vmatrix} = -\Sigma D^{m-2}\xi D \begin{vmatrix} D^{m-2}\eta & D^{m-2}\zeta & \ldots & D^{m-2}\phi \\ D^{m-3}\eta & D^{m-3}\zeta & \ldots & D^{m-3}\phi \\ \vdots & \vdots & & \vdots \\ \eta & \zeta & \ldots & \phi \end{vmatrix}$$

$$=(-)^2\Sigma D^{m-2}\xi\ D\Sigma\ D^{m-3}\eta D \begin{vmatrix} D^{m-3}\zeta & \ldots & D^{m-3}\chi & D^{m-3}\psi \\ D^{m-4}\zeta & \ldots & D^{m-4}\chi & D^{m-4}\psi \\ \vdots & & \vdots & \vdots \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$=(-)^2\Sigma D^{m-2}\xi\ \Sigma\ D^{m-3}\eta\ D^2 \begin{vmatrix} D^{m-3}\zeta & \ldots & D^{m-3}\chi & D^{m-2}\psi \\ D^{m-4}\zeta & \ldots & D^{m-4}\chi & D^{m-4}\psi \\ \vdots & & \vdots & \vdots \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$=(-)^{m-2}\ \Sigma D^{m-2}\xi\ \Sigma D^{m-3}\eta\ \ldots\ \Sigma D\phi\ D^{m-2} \begin{vmatrix} D\chi & D\psi \\ \chi & \psi \end{vmatrix} \quad \ldots \ldots \quad (21.)$$

In the case of the conic of 5-pointic contact this becomes

$$-\Sigma D^3x^2\Sigma D^2y^2\Sigma Dz^2D^3 \begin{vmatrix} 2Dzx & 2Dxy \\ zx & xy \end{vmatrix}$$

I now proceed to the calculation of the conic of 5-pointic contact. This may be effected by any of the three methods indicated above, viz. (1) by means of the symbol □, getting out the factors $(\alpha x+\beta y+\gamma z)^p$; (2) by means of the symbols D; (3) by the reduced formula (21.). As most of the steps have been calculated by two methods, I subjoin some of the leading formulæ and transformations which occur in the process.

Writing for convenience

$$\lambda=\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz},\quad \mu=\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx},\quad \nu=\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy},$$

and putting

$$V=(abcfgh \between xyz)^2,$$

the general expressions (13.) become

$$\begin{aligned}
0&=(abcfgh \between xyz)^2,\\
0&=(abcfgh \between \lambda\mu\nu \between xyz),\\
0&=(abcfgh \between \lambda\mu\nu)^2+(abcfgh \between \Box\lambda\,\Box\mu\,\Box\nu \between xyz),\\
0&=3(abcfgh \between \Box\lambda\,\Box\mu\,\Box\nu \between \lambda\mu\nu)+(abcfgh \between \Box^2\lambda\,\Box^2\mu\,\Box^2\nu \between xyz),\\
0&=3(abcfgh \between \Box\lambda\,\Box\mu\,\Box\nu)^2+4(abcfgh \between \Box^2\lambda\,\Box^2\mu\,\Box^2\nu \between \lambda\mu\nu)+(abcfgh \between \Box^3\lambda\,\Box^3\mu\,\Box^3\nu \between xyz),
\end{aligned}$$

whence

$$a:b:c:f:g:h=$$

$$\left\|\begin{array}{cccccc}
3(\Box\lambda)^2+4\lambda\Box^2\lambda+x\Box^3\lambda & . & . & 6\Box\mu\Box\nu+4(\mu\Box^2\nu+\nu\Box^2\mu)+y\Box^3\nu+z\Box^3\mu & . & . \\
3\lambda\Box\lambda+x\Box^2\lambda & . & . & 3(\mu\Box\nu+\nu\Box\mu)+y\Box^2\nu+z\Box^2\mu & . & . \\
\lambda^2+x\Box\lambda & . & . & 2\mu\nu+y\Box\nu+z\Box\mu & . & . \\
x\lambda & . & . & y\mu+z\nu & . & . \\
x^2 & y^2 & z^2 & 2yz & 2zx & 2xy.
\end{array}\right\| \quad (22.)$$

The most convenient method for developing these formulæ is to begin with forming the 15 minors derived from the last two lines; and by means of them, the 20 minors derived from the last three lines; and so on. It may be noticed that in order to pass to the corresponding expressions in D, *e. g.* D_1, it is only necessary to make $\beta=0$, $\gamma=0$, and to divide out the αs. This gives

$$\lambda=0,\quad \mu=\frac{\partial U}{dz},\quad \nu=-\frac{\partial U}{dy}.$$

The results of the calculations are as follow: the third minors, formed from the two lower lines of (22.), and from the columns the lower constituents of which alone are written down, will be

$$y^2,\, z^2=-yz\frac{\partial U}{dx},\quad z^2,\, x^2=-zx\frac{\partial U}{dy},\quad x^2,\, y^2=-xy\frac{\partial U}{dz},$$

$$2zx,\, 2xy=2x^2\frac{\partial U}{dx},\quad 2xy,\, 2yz=2y^2\frac{\partial U}{dy},\quad 2yz,\, 2zx=2z^2\frac{\partial U}{dz},$$

$$x^2,\, 2yz=x\left(y\frac{\partial U}{dy}-z\frac{\partial U}{dz}\right),\quad x^2,\, 2zx=x^2\frac{\partial U}{dy}\quad x^2,\, 2xy=-x^2\frac{\partial U}{dz},$$

$$y^2,\, 2yz=-y^2\frac{\partial U}{dx},\quad y^2,\, 2zx=y\left(z\frac{\partial U}{dz}-x\frac{\partial U}{dx}\right)\quad y^2,\, 2xy=y^2\frac{\partial U}{dz},$$

$$z^2,\, 2yz=z^2\frac{\partial U}{dx},\quad z^2,\, 2zx=-z^2\frac{\partial U}{dy},\quad z^2,\, 2xy=z\left(x\frac{\partial U}{dx}-y\frac{\partial U}{dy}\right).$$

Similarly, for the second minors,

$$x^2,\ 2xy,\ 2xz = x^3(n-1)^{-2}\mathrm{H},$$

$$2xy,\ y^2\ ,\ 2yz = y^3(n-1)^{-2}\mathrm{H},$$

$$2xz,\ 2yz,\ \ z^2 = z^3(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2yz,\ 2zx = 2\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dz} - zx^2(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2xy,\ 2yz = 2\frac{\partial \mathrm{U}}{dy}\left(\frac{\partial \mathrm{U}}{dz}\right)^2 - yx^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ \ y^2,\ 2zx = 2\left(\frac{\partial \mathrm{U}}{dz}\right)^2\frac{\partial \mathrm{U}}{dx} - xy^2(n-1)^{-2}\mathrm{H},$$

$$2xz,\ \ y^2,\ 2yz = 2\frac{\partial \mathrm{U}}{dz}\left(\frac{\partial \mathrm{U}}{dx}\right)^2 - zy^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ 2yz,\ \ z^2 = 2\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dy} - yz^2(n-1)^{-2}\mathrm{H},$$

$$2zx,\ 2xy,\ \ z^2 = 2\frac{\partial \mathrm{U}}{dx}\left(\frac{\partial \mathrm{U}}{dy}\right)^2 - xz^2(n-1)^{-2}\mathrm{H},$$

$$y^2,\ \ z^2,\ 2yz = \left(\frac{\partial \mathrm{U}}{dx}\right)^3,$$

$$y^2,\ \ z^2,\ 2zx = -\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dy} + yz^2(n-1)^{-2}\mathrm{H}\right\},$$

$$y^2,\ \ z^2,\ 2xy = -\left\{\left(\frac{\partial \mathrm{U}}{dx}\right)^2\frac{\partial \mathrm{U}}{dz} + y^2z(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ \ x^2,\ 2yz = -\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dx} + z^2x(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ \ x^2,\ 2zx = \left(\frac{\partial \mathrm{U}}{dy}\right)^3,$$

$$z^2,\ \ x^2,\ 2xy = -\left\{\left(\frac{\partial \mathrm{U}}{dy}\right)^2\frac{\partial \mathrm{U}}{dz} + zx^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2yz = -\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2\frac{\partial \mathrm{U}}{dx} + xy^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2zx = -\left\{\left(\frac{\partial \mathrm{U}}{dz}\right)^2\frac{\partial \mathrm{U}}{dy} + x^2y(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2xy = \left(\frac{\partial \mathrm{U}}{dz}\right)^3,$$

$$x^2,\ \ y^2,\ z^2 = \frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} - xyz(n-1)^{-2}\mathrm{H},$$

$$2yz,\ 2zx,\ 2xy = -2\frac{\partial \mathrm{U}}{dx}\frac{\partial \mathrm{U}}{dy}\frac{\partial \mathrm{U}}{dz} - 2xyz(n-1)^{-2}\mathrm{H}.$$

Similarly, for the first minors,

$$x^2,\ y^2,\ z^2,\ 2yz=\frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\right)\right\},$$

$$x^2,\ y^2,\ z^2,\ 2zx=\frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right\},$$

$$x^2,\ y^2,\ z^2,\ 2xy=\frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\right)\right\},$$

$$y^2,\ z^2,\ 2zx,\ 2xy=\frac{\partial U}{dx}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right.$$
$$\left.-\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)+3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\right)\right\}-3yz\frac{H^2}{(n-1)^2},$$

$$y^2,\ z^2,\ 2xy,\ 2yz=\frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dx^2}-\frac{\partial U}{dx}\frac{\partial^2 U}{dxdz}\right)\right\},$$

$$y^2,\ z^2,\ 2zx,\ 2yz=\frac{\partial U}{dx}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dxdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dx^2}\right)\right\},$$

$$z^2,\ x^2,\ 2xy,\ 2zx=\frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dydz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dy^2}\right)\right\},$$

$$z^2,\ x^2,\ 2xy,\ 2yz=\frac{\partial U}{dy}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\right)\right.$$
$$\left.-\frac{\partial U}{dx}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)+3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\right)\right\}-3zx\frac{H^2}{(n-1)^2},$$

$$z^2,\ x^2,\ 2yz,\ 2zx=\frac{\partial U}{dy}\left\{\frac{\partial U}{dy}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\right)-3H\left(\frac{\partial U}{dx}\frac{\partial^2 U}{dy^2}-\frac{\partial U}{dy}\frac{\partial^2 U}{dydx}\right)\right\},$$

$$x^2,\ y^2,\ 2zx,\ 2xy=\frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dz^2}-\frac{\partial U}{dz}\frac{\partial^2 U}{dzdy}\right)\right\},$$

$$x^2,\ y^2,\ 2yz,\ 2xy=\frac{\partial U}{dz}\left\{\frac{\partial U}{dz}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dzdx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dz^2}\right)\right\},$$

$$x^2,\ y^2,\ 2yz,\ 2zx=\frac{\partial U}{dz}\left\{\frac{\partial U}{dx}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\right)\right.$$
$$\left.-\frac{\partial U}{dy}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)+3H\left(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\right)\right\}-3xy\frac{H^2}{(n-1^2)},$$

$$2yz,\ 2zx,\ 2xy,\ x^2=2\frac{\partial U}{dy}\frac{\partial U}{dz}\left(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\right)-3H\left\{\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dz^2}-\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dy^2}\right\},$$

$$2yz,\ 2zx,\ 2xy,\ y^2=2\frac{\partial U}{dz}\frac{\partial U}{dx}\left(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\right)-3H\left\{\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dx^2}-\left(\frac{\partial U}{dx}\right)^2\frac{\partial^2 U}{dz^2}\right\},$$

$$2yz,\ 2zx,\ 2xy,\ z^2=2\frac{\partial U}{dx}\frac{\partial U}{dy}\left(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dz}\right)-3H\left\{\left(\frac{\partial U}{dx}\right)^2\frac{\partial^2 U}{dy^2}-\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dx^2}\right\}.$$

Proceeding to the calculation of the final expressions for the coefficients a, b, c, f, g, h of the conic of five-pointed contact, which is given rather more in detail, we have, writing $\frac{d\mathrm{U}}{dx}=u$, $\frac{d\mathrm{U}}{dy}=v$, $\frac{d\mathrm{U}}{dz}=w$,

$$f=x^2, y^2, z^2, 2zx, 2xy=\mathrm{D}^4x^2(y^2, z^2, 2zx, 2xy)$$

$$+\mathrm{D}^4y^2(z^2, 2zx, 2xy, x^2)$$

$$+\mathrm{D}^4z^2(2zx, 2xy, x^2, y^2)$$

$$+2\mathrm{D}^4zx(2xy, x^2, y^2, z^2)$$

$$+2\mathrm{D}^4xy(x^2, y^2, z^2, 2zx)$$

$$=-(3(\mathrm{D}w)^2+4w\mathrm{D}^2w-y\mathrm{D}^3w)v\,(v\,\mathrm{D}\,\mathrm{H}-3\mathrm{HD}\,v\,)$$

$$+(3(\mathrm{D}v\,)^2+4v\,\mathrm{D}^2v\,+z\mathrm{D}^3v\,)w(w\mathrm{D}\,\mathrm{H}-3\mathrm{HD}\,w)$$

$$-(\qquad x\mathrm{D}^3v\,)w(w\mathrm{D}_2\mathrm{H}-3\mathrm{HD}_2w)$$

$$-(\qquad x\mathrm{D}^3w)v\,(v\,\mathrm{D}_1\mathrm{H}-3\mathrm{HD}_1w)$$

$$=3(w^2(\mathrm{D}v)^2-v^2(\mathrm{D}w)^2)\mathrm{DH}+4vw(w\mathrm{D}^2v-v\mathrm{D}^2w)\mathrm{DH}$$

$$+(w^2z\mathrm{D}^3v+v^2y\mathrm{D}^3w)\mathrm{DH}-x(w^2\mathrm{D}^3v\mathrm{D}_2\mathrm{H}+v^2\mathrm{D}^3w\mathrm{D}_1\mathrm{H})$$

$$+3\mathrm{H}\{3\mathrm{D}v\mathrm{D}w(v\mathrm{D}w-w\mathrm{D}v)+4vw(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v)$$

$$-yv\mathrm{D}v\mathrm{D}^3w-zw\mathrm{D}w\mathrm{D}^3v+xv\mathrm{D}_1v\mathrm{D}^3w+xw\mathrm{D}_2w\mathrm{D}^3v\}$$

$$=3(w\mathrm{D}v-v\mathrm{D}w)\mathrm{D}(vw)\mathrm{DH}+4vw\mathrm{D}(w\mathrm{D}v-v\mathrm{D}w)\mathrm{DH}$$

$$-9(w\mathrm{D}v-v\mathrm{D}w)\mathrm{D}v\mathrm{D}w\mathrm{H}+12vw(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v)\mathrm{H}$$

$$+\mathrm{D}^3v\{w^2(z\mathrm{DH}-x\mathrm{D}_2\mathrm{H})-3\mathrm{H}w(z\mathrm{D}w-x\mathrm{D}_2w)\}$$

$$+\mathrm{D}^3w\{v^2(y\mathrm{DH}-x\mathrm{D}_1\mathrm{H})-3\mathrm{H}v(y\mathrm{D}v-x\mathrm{D}_1v)\}$$

$$=(n-1)^{-2}x^2\{3\mathrm{D}(vw)\mathrm{HDH}+4vw(\mathrm{DH})^2-9\mathrm{D}v\mathrm{D}w\mathrm{H}^2\}$$

$$+12vw\mathrm{H}(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v)$$

$$+\;3vw\mathrm{HD}^2(v\mathrm{D}w-w\mathrm{D}v)$$

$$-\;3vw\mathrm{H}(\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v),$$

which, omitting the common factor $(n-1)^{-2}x^2$,

$$=3\mathrm{D}(vw)\mathrm{HDH}+4vw(\mathrm{DH})^2+9(n-1)^{-2}x^2\mathrm{H}^3\frac{\partial^2\mathrm{U}}{dydz}+9vw\mathfrak{A}\mathrm{H}^2$$

$$+9\frac{3(n-2)}{n-1}vw\mathfrak{A}\mathrm{H}^2-9\frac{x}{n-1}vw\mathrm{H}\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{H}\frac{\partial\mathrm{H}}{dy}+\mathfrak{G}\frac{\partial\mathrm{H}}{dz}\right)-3vw\mathrm{HD}^2\mathrm{H}.$$

But

$$3\mathrm{D}(vw)\mathrm{HDH}+6\,\frac{3(n-2)}{n-1}vw\mathfrak{A}\mathrm{H}^2-6\,\frac{x}{n-1}vw\mathrm{H}\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{H}\frac{\partial\mathrm{H}}{dy}+\mathfrak{G}\frac{\partial\mathrm{H}}{dz}\right)$$

$$=3(v\mathrm{D}w+w\mathrm{D}v)\left(v\frac{\partial\mathrm{H}}{dz}-w\frac{\partial\mathrm{H}}{dy}\right)\mathrm{H}+6vw\left(\mathrm{D}w\frac{\partial\mathrm{H}}{dy}-\mathrm{D}v\frac{\partial\mathrm{H}}{dz}\right)\mathrm{H}$$

$$=(3v^2\mathrm{D}w+3vw\mathrm{D}v-6vw\mathrm{D}v)\mathrm{H}\frac{\partial\mathrm{H}}{dz}-(3vw\mathrm{D}w+3w^2\mathrm{D}v-6vw\mathrm{D}w)\mathrm{H}\frac{\partial\mathrm{H}}{dy}$$

$$=3\mathrm{H}v(v\mathrm{D}w-w\mathrm{D}v)\frac{\partial\mathrm{H}}{dz}-3\mathrm{H}w(w\mathrm{D}v-v\mathrm{D}w)\frac{\partial\mathrm{H}}{dy}.$$

$$=-3(n-1)^{-2}x^2\mathrm{H}^2\left(w\frac{\partial\mathrm{H}}{dy}+v\frac{\partial\mathrm{H}}{dz}\right).$$

Hence, excepting the common factor $(n-1)^{-2}x^2$, the total expression for f

$$=9(n-1)^{-2}x^2\mathrm{H}^3\frac{\partial^2\mathrm{U}}{dydz}-3(n-1)^{-2}x^2\mathrm{H}^2\left(\frac{\partial\mathrm{U}}{dy}\frac{\partial\mathrm{H}}{dz}+\frac{\partial\mathrm{U}}{dz}\frac{\partial\mathrm{H}}{dy}\right)$$

$$-vw\left\{3\mathrm{H}\left(\mathrm{D}^2\mathrm{H}+\frac{x}{n-1}\left(\mathfrak{A}\frac{\partial\mathrm{H}}{dx}+\mathfrak{H}\frac{\partial\mathrm{H}}{dy}+\mathfrak{G}\frac{\partial\mathrm{H}}{dz}\right)-\frac{3(n-2)}{n-1}\mathrm{H}\mathfrak{A}\right)-4(\mathrm{DH})^2\right\}.$$

And comparing this with Mr. CAYLEY's expressions in arts. 14 and 15 of his memoir "On the Conic of Five-pointic Contact," in the Philosophical Transactions for 1859, pp. 376, 377; and bearing in mind that in his formulæ we must make $\lambda=1$, $\mu=0$, $\nu=0$, in order to institute a comparison; and lastly, dividing throughout by $(n-1)^{-2}x^29\mathrm{H}^3$, and multiplying by 2, the expression above written becomes

$$2\frac{\partial^2\mathrm{U}}{dydz}-\frac{2}{3}\frac{1}{\mathrm{H}}\left(\frac{\partial\mathrm{H}}{dy}\frac{\partial\mathrm{U}}{dz}+\frac{\partial\mathrm{H}}{dz}\frac{\partial\mathrm{U}}{dy}\right)-2\Lambda\frac{\partial\mathrm{U}}{dy}\frac{\partial\mathrm{U}}{dz},$$

which is in fact the coefficient of YZ in his general formula, viz.

$$\left(\mathrm{X}\frac{\partial}{dx}+\mathrm{Y}\frac{\partial}{dy}+\mathrm{Z}\frac{\partial}{dz}\right)^2\mathrm{U}-\left(\frac{2}{3}\frac{1}{\mathrm{H}}\left(\mathrm{X}\frac{\partial\mathrm{H}}{dx}+\mathrm{Y}\frac{\partial\mathrm{H}}{dy}+\mathrm{Z}\frac{\partial\mathrm{H}}{dz}\right)+\Lambda\left(\mathrm{X}\frac{\partial\mathrm{U}}{dx}+\mathrm{Y}\frac{\partial\mathrm{U}}{dy}+\mathrm{Z}\frac{\partial\mathrm{U}}{dz}\right)\right)$$
$$\times\left(\mathrm{X}\frac{\partial\mathrm{U}}{dx}+\mathrm{Y}\frac{\partial\mathrm{U}}{dy}+\mathrm{Z}\frac{\partial\mathrm{U}}{dz}\right)=0.$$

The identity of the expressions for f, as deduced by the present method, with that deduced by Mr. CAYLEY having been thus demonstrated, it is unnecessary to pursue the calculations further.

The following are the principal subsidiary formulæ used in the foregoing calculations.

$$\delta=\alpha x+\beta y+\gamma z$$

$$y\nu-z\mu=\delta\frac{\partial\mathrm{U}}{dx},\quad z\lambda-x\nu=\delta\frac{\partial\mathrm{U}}{dy},\quad x\mu-y\lambda=\delta\frac{\partial\mathrm{U}}{dz}$$

$$y\square\nu-z\square\mu=\delta\square\frac{\partial\mathrm{U}}{dx},\quad z\square\lambda-x\square\nu=\delta\square\frac{\partial\mathrm{U}}{dy},\quad x\square\mu-y\square\lambda=\delta\square\frac{\partial\mathrm{U}}{dz}$$

$$x\square\frac{\partial\mathrm{U}}{dx}+y\square\frac{\partial\mathrm{U}}{dy}+z\square\frac{\partial\mathrm{U}}{dz}=0.$$

Also writing

$$\square\lambda\frac{\partial U}{dx}+\square\mu\frac{\partial U}{dy}+\square\nu\frac{\partial U}{dz}=\Pi,$$

we have

$$\mu\square\nu-\nu\square\mu=\alpha\Pi,\ \nu\square\lambda-\lambda\square\nu=\beta\Pi,\ \lambda\square\mu-\mu\square\lambda=\gamma\Pi;$$

also

$$-\Pi=\begin{vmatrix} \frac{\partial^2 U}{dx} & \frac{\partial^2 U}{dxdy} & \frac{\partial^2 U}{dxdz} & \frac{\partial U}{dx} & \alpha \\ \frac{\partial^2 U}{dydx} & \frac{\partial^2 U}{dy^2} & \frac{\partial^2 U}{dydz} & \frac{\partial U}{dy} & \beta \\ \frac{\partial^2 U}{dzdx} & \frac{\partial^2 U}{dzdy} & \frac{\partial^2 U}{dz^2} & \frac{\partial U}{dz} & \gamma \\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz} & 0 & 0 \\ \alpha & \beta & \gamma & 0 & 0 \end{vmatrix}=-\frac{\delta^2}{(n-1)^2}H,$$

where H is the Hessian of U. The corresponding reduced form of this expression, *i. e.* when $\alpha=1$, $\beta=0$, $\gamma=0$, is

$$\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dy^2}-2\frac{\partial U}{dz}\frac{\partial U}{dy}\frac{\partial^2 U}{dydz}+\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dz^2}=-\frac{x^2}{(n-1)^2}H.$$

And by means of these we may deduce the following:—

$$wDv-vDw=(n-1)^{-2}x^2H,\ wD_1v-vD_1u=(n-1)^{-2}yxH,\ wD_2v-vD_2w=(n-1)^{-2}zxH,$$
$$uDw-wDu=(n-1)^{-2}xyH,\ uD_1w-wD_1u=(n-1)^{-2}y^2H,\ uD_2w-wD_2u=(n-1)^{-2}zyH,$$
$$vDu-uDv=(n-1)^{-2}xzH,\ vD_1u-uD_1v=(n-1)^{-2}yzH,\ vD_2u-uD_2v=(n-1)^{-2}z^2H.$$

Again,

$$yD_2H-zD_1H=3(n-2)\frac{\partial U}{dx}H,\ zDH-xD_2H=3(n-2)\frac{\partial U}{dy}H,\ xD_1H-yDH=3(n-2)\frac{\partial U}{dz}H$$

$$yD_2u-zD_1u=(n-1)\left(\frac{\partial U}{dx}\right)^2,\ zDw-xD_2w=xD_1v-yDv=(n-1)\frac{\partial U}{dy}\frac{\partial U}{dz}.$$

To which may be added,

$$\square\lambda D\,u+\square\mu D\,v+\square\nu D\,w=\delta H\lambda,$$
$$\square\lambda D_1u+\square\mu D_1v+\square\nu D_1w=\delta H\mu,$$
$$\square\lambda D_2u+\square\mu D_2v+\square\nu D_2w=\delta H\nu,$$
$$uDv+vD_1v+wD_2v=0,$$

$$D^2u=-(n-1)^{-2}\frac{\partial}{dx}(x^2H)-3Au+3xH,$$

$$D^2v=-(n-1)^{-2}\frac{\partial}{dy}(x^2H)-3Av,$$

$$D^2w=-(n-1)^{-2}\frac{\partial}{dz}(x^2H)-3Aw,$$

$$uD^2v \quad +vDD_1v \; +wDD_2v \qquad = -(n-1)^{-1}vxH,$$
$$uD^2w \quad +vDD_1w + wDD_2w \qquad = -(n-1)^{-1}wxH,$$
$$xDD_1u + yDD_1v \; + zDD_1w \qquad = \quad (n-1)^{-2}xyH,$$
$$yDv \quad - xD_1v \quad = xD_2w - zDw = -(n-1) \;\; vw,$$
$$vDD_2w - wDD_2u = \quad D\{(n-1)^{-2}yzH\} - (n-1)^{-1}vyH,$$
$$uDD_1u - uDD_1v = \quad D\{(n-1)^{-1}yzH\} - (n-1)^{-1}wzH,$$
$$zD_1u \quad - yD_2u \quad = - \quad (n-1)^{-2}u^2,$$

$$DvDw = -(n-1)^{-2}x^2H\frac{\partial U}{dydz} - vw\mathfrak{A},$$

$$DvD^2w - DwD^2v = (n-1)^{-2}x^2\left\{Dw\frac{\partial H}{dy} - Dv\frac{\partial H}{dz} - \mathfrak{A}H\right\},$$

$$Dw\frac{\partial H}{dy} - Dv\frac{\partial H}{dz} = (n-1)^{-1}\left\{3(n-2)\mathfrak{A}H - x\left(\mathfrak{A}\frac{\partial H}{dx} + \mathfrak{H}\frac{\partial H}{dy} + \mathfrak{G}\frac{\partial H}{dz}\right)\right\},$$

whence

$$DvD^2w - DwD^2v = (n-1)^{-2}x^2\left\{\left(\frac{3(n-2)}{n-1} - 1\right)\mathfrak{A}H - \frac{x}{n-1}\left(\mathfrak{A}\frac{\partial H}{dx} + \mathfrak{H}\frac{\partial H}{dy} + \mathfrak{G}\frac{\partial H}{dz}\right)\right\},$$

to which many others might be added.